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# The Friedmann-Lemaître-Robertson-Walker Metric

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## Abstract

After stating the formal requirements, the concept of a metric is gradually explained and illustrated starting from simple cases in plane geometry, working the way up to the derivation of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, being the simplest metric for a homogeneous and isotropic spacetime. It is then briefly outlined how it relates to Einstein's field equations of general relativity, giving rise to Friedmann's solution which describes the evolution of a homogeneous and isotropic universe.

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## 1 Definition of a metric

A metric is a distance function which gives the separation between 2 arbitrary points of a given set of points. It needs to satisfy 4 criteria to formally qualify as a metric:

1. The distance between any 2 points is always positive.
2. The distance between 2 points is zero if and only if the 2 points are the same.
3. The distance between A and B equals the distance between B and A.
4. The distance between A and B is less or equal than the distance between A and C plus the distance between C and B.

The set of points, together with the metric, form a metric space. Metrics are often used in the context of points characterized by geometrical coordinates but also exist for more exotic sets. Consider the set of 3-letter words  $\{ \text{cat, car, dog, ...} \}$ . The function defining the separation between 2 words as the number of letters that need to be changed to transition from one word to another, is a valid metric.

If a metric expresses the infinitesimal distance between 2 points, the distance along an arbitrary path connecting 2 widely separated points is obtained by integrating the metric.

## 2 Surface metrics

### 2.1 Curvature dependent angular term

#### 2.1.1 Flat surfaces

Consider the points  $M$  and  $N$  with cartesian coordinates  $M(x, y)$  and  $N(x + dx, y + dy)$  on a plane surface as illustrated in figure 1.

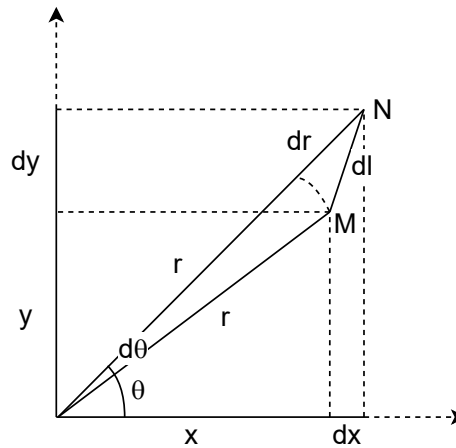


Figure 1: Infinitesimal distance  $dl$  on a plane surface between points  $M$  and  $N$  with cartesian coordinates  $M(x, y)$  and  $N(x + dx, y + dy)$  and polar coordinates  $M(r, \theta)$  and  $N(r + dr, \theta + d\theta)$ .

The distance  $dl$  between  $M$  and  $N$  is given by Pythagoras' theorem as:

$$dl^2 = dx^2 + dy^2$$

When polar coordinates  $M(r, \theta)$  and  $N(r + dr, \theta + d\theta)$  are used instead of cartesian, the transformation is governed by:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{matrix} (1a) \\ (1b) \end{matrix}$$

Deriving equations (1a) and (1b) leads to:

$$\begin{cases} dx = dr \cos \theta - r \sin \theta d\theta \\ dy = dr \sin \theta + r \cos \theta d\theta \end{cases}$$

Taking the square of both equations and summing them together gives:

$$\begin{aligned} dx^2 + dy^2 &= (dr \cos \theta - r \sin \theta d\theta)^2 + (dr \sin \theta + r \cos \theta d\theta)^2 \\ dx^2 + dy^2 &= dr^2 \cos^2 \theta + r^2 \sin^2 \theta d\theta^2 - 2 dr \cos \theta r \sin \theta d\theta \\ &\quad + dr^2 \sin^2 \theta + r^2 \cos^2 \theta d\theta^2 + 2 dr \sin \theta r \cos \theta d\theta \\ dx^2 + dy^2 &= dr^2 \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1} + r^2 d\theta^2 \underbrace{(\sin^2 \theta + \cos^2 \theta)}_{=1} \\ dx^2 + dy^2 &= dr^2 + r^2 d\theta^2 \end{aligned}$$

The distance  $dl$  between  $M$  and  $N$  in polar coordinates is therefore:

$$dl^2 = dr^2 + r^2 d\theta^2 \quad (2)$$

This result is not unexpected if one takes into account that the length  $dr$  and the arc  $r d\theta$  converge to the sides of a rectangular triangle in the limit where  $dl$  tends to zero.

Expression (2) is a metric for a flat surface.

### 2.1.2 Spherical surfaces

Now consider the points  $M$  and  $N$  with polar coordinates  $M(r, \theta)$  and  $N(r + dr, \theta + d\theta)$  on the surface of a sphere with radius  $R$  as shown in figure 2.

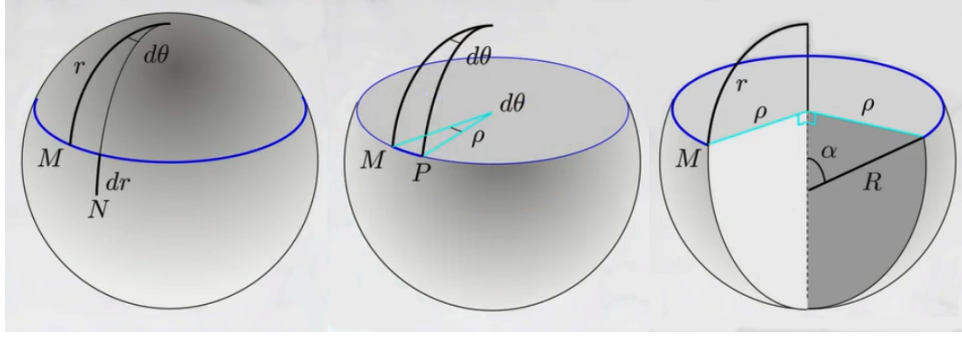


Figure 2: Infinitesimal distance on a spherical surface between points M and N with polar coordinates  $M(r, \theta)$  and  $N(r + dr, \theta + d\theta)$ . (Credit: Richard Taillet)

The distance  $dl$  between  $M$  and  $N$  in polar coordinates is then:

$$dl^2 = dr^2 + \rho^2 d\theta^2 \quad (3)$$

Coordinate  $\rho$  is a function of radius  $R$  and angle  $\alpha$ , of which the latter in turn is a function of coordinate  $r$  and radius  $R$ :

$$\rho = R \sin \alpha$$

$$\alpha = \frac{r}{R}$$

Substitution in equation (3) yields:

$$dl^2 = dr^2 + R^2 \sin^2 \left( \frac{r}{R} \right) d\theta^2 \quad (4)$$

Expression (4) is a metric for a spherical surface.

### 2.1.3 Hyperbolic surfaces

Without formally deriving it, the metric for a hyperbolic surface is very similar to the one for a spherical surface. The only difference is the sine which has to be substituted by the hyperbolic sine:

$$dl^2 = dr^2 + R^2 \sinh^2 \left( \frac{r}{R} \right) d\theta^2 \quad (5)$$

### 2.1.4 Curved surfaces

The metrics (2), (4), (5) derived in the previous sections are elegantly combinable in a single metric by introducing the curvature parameter  $\kappa$  which takes the value +1, 0 or -1 depending on whether the surface is spherical, flat or hyperbolic.

$$dl^2 = dr^2 + S_\kappa^2(r) d\theta^2$$

$$\text{with } S_\kappa(r) = \begin{cases} R \sin\left(\frac{r}{R}\right) & \text{when } \kappa = +1 \\ r & \text{when } \kappa = 0 \\ R \sinh\left(\frac{r}{R}\right) & \text{when } \kappa = -1 \end{cases}$$

Other properties of spherical, flat and hyperbolic geometries are summarized in table 1. More notably the sum of the angles  $\Sigma\alpha$  of a triangle and the circumference  $c$  of a circle are essentially different than what is the case in flat geometries.

Curvature	Geometry	Triangle	Circle
$\kappa = +1$	spherical	$\Sigma\alpha > \pi$	$c < 2\pi r$
$\kappa = 0$	flat	$\Sigma\alpha = \pi$	$c = 2\pi r$
$\kappa = -1$	hyperbolic	$\Sigma\alpha < \pi$	$c > 2\pi r$

Table 1: Classification of geometries depending the value of curvature parameter  $\kappa$ .

While 3-dimensional hyperbolic or spherical geometries are not easy to visualize, their 2-dimensional counterparts as depicted in figure 3, help to understand the essential differences.

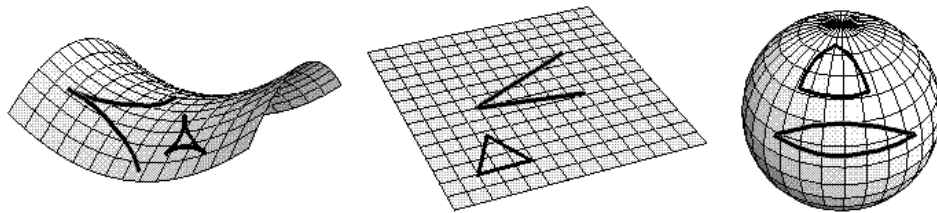


Figure 3: Visualization of hyperbolic, flat and spherical 2-dimensional geometries.

## 2.2 Curvature dependent distance term

### 2.2.1 Spherical surfaces

An alternative metric for a spherical surface is relatively easy to derive by considering the surface as contained in a 3-dimensional space. If the positions of  $M$  and  $N$  are then characterized by cartesian coordinates  $M(x, y, z)$  and  $N(x + dx, y + dy, z + dz)$ , the distance  $dl$  between them is:

$$dl^2 = dx^2 + dy^2 + dz^2 \quad (6)$$

On the other hand, any point on the surface of a sphere with radius  $R$  must satisfy the condition:

$$x^2 + y^2 + z^2 = R^2 \quad (7)$$

The relationship between the cartesian coordinates  $(x, y)$  in the XY plane and the equivalent polar coordinates  $(\rho, \theta)$  in the XY plane are given by equations (1a) and (1b) but now with  $\rho$  as radial coordinate instead of  $r$ :

$$\begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases} \quad (8a)$$

$$\quad \quad \quad (8b)$$

Deriving equations (8a) and (8b) and summing the squares of the derivatives results in:

$$dx^2 + dy^2 = d\rho^2 + \rho^2 d\theta^2 \quad (9)$$

Summing the squares of equations (8a) and (8b) yields:

$$\begin{aligned} x^2 + y^2 &= \rho^2 \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1} \\ x^2 + y^2 &= \rho^2 \end{aligned} \quad (10)$$

Combining equations (6) and (9) gives:

$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 \quad (11)$$

Combining equations (7) and (10) leads to:

$$z^2 = R^2 - \rho^2 \quad (12)$$

Deriving equation (12) results in:

$$2 z dz = -2 \rho d\rho$$

$$dz = -\frac{\rho d\rho}{z} \quad (13)$$

Substituting equations (13) and (12) in equation (11) yields:

$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$$

$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + \frac{\rho^2 d\rho^2}{z^2}$$

$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + \frac{\rho^2 d\rho^2}{R^2 - \rho^2}$$

$$dl^2 = \frac{R^2}{R^2 - \rho^2} d\rho^2 + \rho^2 d\theta^2 \quad (14)$$

Note that  $\rho$  is a coordinate which is unmeasurable on the surface of the sphere.

Although it looks different, the metric of equation (14) is fully equivalent with the metric of equation (4). Their difference in appearance is solely the result of the use of different coordinate systems.

### 2.2.2 Hyperbolic surfaces

Without formally deriving it, the metric for a hyperbolic surface is again very similar to the one for a spherical surface. The only difference this time is the sign in the denominator:

$$dl^2 = \frac{R^2}{R^2 + \rho^2} d\rho^2 + \rho^2 d\theta^2 \quad (15)$$

### 2.2.3 Curved surfaces

The metrics (2), (14), (15) derived in the previous sections are again elegantly combinable in a single metric using the curvature parameter  $\kappa$  which takes the value +1, 0 or -1 depending on whether the surface is spherical, flat or hyperbolic:

$$dl^2 = \frac{R^2}{R^2 - \kappa \rho^2} d\rho^2 + \rho^2 d\theta^2$$

### 3 Spatial metrics

#### 3.1 Curvature dependent distance term

As was the case for a spherical surface, the metric for a spherically curved 3-dimensional space is relatively easy to derive by considering it as contained in a one dimension higher space. If the positions of  $M$  and  $N$  are then characterized by cartesian coordinates  $M(x, y, z, u)$  and  $N(x + dx, y + dy, z + dz, u + du)$ , the distance  $dl$  between them is:

$$dl^2 = dx^2 + dy^2 + dz^2 + du^2 \quad (16)$$

On the other hand, any point must satisfy the following condition involving the radius of curvature  $R$ :

$$x^2 + y^2 + z^2 + u^2 = R^2 \quad (17)$$

The relationship between the cartesian coordinates  $(x, y, z)$  in the XYZ subspace and the equivalent spherical coordinates  $(\rho, \theta, \phi)$ <sup>1</sup> is illustrated in figure 4.

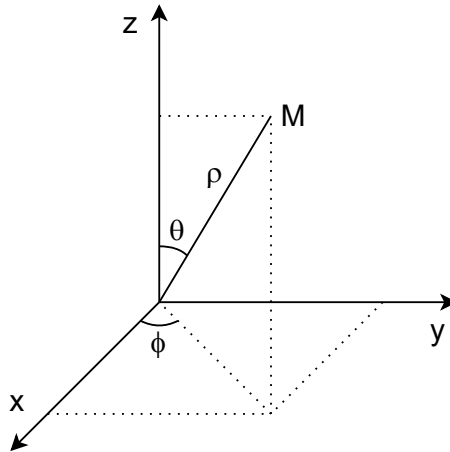


Figure 4: Spherical coordinates  $(\rho, \theta, \phi)$  in relation to cartesian coordinates  $(x, y, z)$   
In physics,  $\theta$  usually denotes the polar angle and  $\phi$  the azimuthal angle.  
In mathematics, the meaning of both angles is often swapped.

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<sup>1</sup>The physics convention is used in which  $\theta$  is the polar angle in the range  $[0, \pi]$  and  $\phi$  the azimuthal angle in the range  $[0, 2\pi]$ . Mathematical textbooks often switch the meaning of  $\theta$  and  $\phi$ .

Mathematically, the transformation is expressed by a set of 3 equations:

$$\begin{cases} x = \rho \sin \theta \cos \phi & (18a) \\ y = \rho \sin \theta \sin \phi & (18b) \\ z = \rho \cos \theta & (18c) \end{cases}$$

Deriving equations (18a), (18b) and (18c) and summing the squares results in:

$$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 \quad (19)$$

Similarly, the sum of the squares of equations (18a), (18b) and (18c) gives:

$$x^2 + y^2 + z^2 = \rho^2 \quad (20)$$

Combining equations (16) and (19) yields:

$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 + du^2 \quad (21)$$

Combining equations (17) and (20) leads to:

$$u^2 = R^2 - \rho^2 \quad (22)$$

Deriving equation (22) results in:

$$\begin{aligned} 2 u du &= -2 \rho d\rho \\ du &= -\frac{\rho d\rho}{u} \end{aligned} \quad (23)$$

Substituting equations (23) and (22) in equation (21) yields:

$$\begin{aligned} dl^2 &= d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 + du^2 \\ dl^2 &= d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 + \frac{\rho^2 d\rho^2}{u^2} \\ dl^2 &= d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 + \frac{\rho^2 d\rho^2}{R^2 - \rho^2} \\ dl^2 &= \frac{R^2}{R^2 - \rho^2} d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \end{aligned} \quad (24)$$

Note that also here,  $\rho$  is a coordinate which is unmeasurable in the 3-dimensional space.

Similar to surfaces, the generalized form of metric (24) requires a curvature parameter  $\kappa$  which takes the value +1, 0 or -1 depending on whether the space has a spherical, flat or hyperbolic curvature:

$$dl^2 = \frac{R^2}{R^2 - \kappa \rho^2} d\rho^2 + \rho^2 d\Omega^2$$

with  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$

### 3.2 Curvature dependent angular term

Define the angle  $\alpha$  so that:

$$\begin{cases} \rho = R \sin \alpha \\ \alpha = \frac{r}{R} \end{cases} \quad (25a)$$

$$\quad \quad \quad (25b)$$

Deriving equation (25a) results in:

$$d\rho = -R \cos \alpha d\alpha \quad (26)$$

Using equations (25a) and (26), the first term of equation (24) becomes:

$$\begin{aligned} \frac{R^2}{R^2 - \rho^2} d\rho^2 &= \frac{R^2 R^2 \cos^2 \alpha d\alpha^2}{R^2 - R^2 \sin^2 \alpha} \\ \frac{R^2}{R^2 - \rho^2} d\rho^2 &= \frac{R^2 \cos^2 \alpha d\alpha^2}{\underbrace{1 - \sin^2 \alpha}_{=\cos^2 \alpha}} \\ \frac{R^2}{R^2 - \rho^2} d\rho^2 &= R^2 d\alpha^2 \end{aligned} \quad (27)$$

Substituting equations (25a) and (27) in equation (24) yields:

$$dl^2 = R^2 d\alpha^2 + R^2 \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\phi^2)$$

Making use of equation (25b) leads to:

$$dl^2 = dr^2 + R^2 \sin^2 \left( \frac{r}{R} \right) (d\theta^2 + \sin^2 \theta d\phi^2) \quad (28)$$

The generalized form of metric (28) again requires a curvature parameter  $\kappa$  which takes the value +1, 0 or -1 depending on whether the space has a spherical, flat or hyperbolic curvature:

$$\begin{aligned}
 dl^2 &= dr^2 + S_\kappa^2(r) d\Omega^2 \\
 \text{with } d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 \\
 \text{and } S_\kappa(r) &= \begin{cases} R \sin\left(\frac{r}{R}\right) & \text{when } \kappa = +1 \\ r & \text{when } \kappa = 0 \\ R \sinh\left(\frac{r}{R}\right) & \text{when } \kappa = -1 \end{cases}
 \end{aligned} \tag{29}$$

## 4 Spacetime metrics

### 4.1 The scale factor

Up to now, the radius of curvature  $R$  has been considered constant. However, nothing prohibits that  $R$  is a function of time  $R(t)$ .

Separating  $R^2(t)$  out of equation (29) gives:

$$dl^2 = R^2(t) \left[ d\left(\frac{r}{R(t)}\right)^2 + \frac{S_\kappa^2(r)}{R^2(t)} d\Omega^2 \right]$$

Using  $\alpha = r/R$  from equation (25b) this becomes:

$$\begin{aligned}
 dl^2 &= R^2(t) \underbrace{\left[ d\alpha^2 + S_\kappa^2(\alpha) d\Omega^2 \right]}_{\text{comoving coordinates}} \\
 \text{with } d\Omega^2 &= d\theta^2 + \sin^2\theta d\phi^2 \\
 \text{and } S_\kappa(\alpha) &= \begin{cases} \sin \alpha & \text{when } \kappa = +1 \\ \alpha & \text{when } \kappa = 0 \\ \sinh \alpha & \text{when } \kappa = -1 \end{cases}
 \end{aligned} \tag{30}$$

As illustrated in figure 2 for the 2D analogy,  $r$  scales proportionally with  $R$  so that  $\alpha$  remains constant. The part between square brackets in equation (30) is then solely expressed in angular coordinates which remain constant as  $R(t)$  evolves with time. Such coordinates are called comoving coordinates.

It is convenient to define a dimensionless scale factor  $a(t)$  as the ratio between the radius of curvature  $R(t)$  and its present value  $R(t_0) = R_0$ :

$$a(t) \equiv \frac{R(t)}{R_0}$$

Introducing the scale factor, equation (30) transforms into:

$$dl^2 = a^2(t) \left[ d(R_0 \alpha)^2 + S_\kappa^2(\alpha) d\Omega^2 \right]$$

$$\text{with } d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

$$\text{and } S_\kappa(\alpha) = \begin{cases} R_0 \sin \alpha & \text{when } \kappa = +1 \\ R_0 \alpha & \text{when } \kappa = 0 \\ R_0 \sinh \alpha & \text{when } \kappa = -1 \end{cases}$$

Transitioning from the comoving coordinate  $\alpha$  to another comoving coordinate  $r' = R_0 \alpha$  gives:

$$dl^2 = a^2(t) \left[ dr'^2 + S_\kappa^2(r') d\Omega^2 \right]$$

$$\text{with } d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

$$\text{and } S_\kappa(r') = \begin{cases} R_0 \sin \left( \frac{r'}{R_0} \right) & \text{when } \kappa = +1 \\ r' & \text{when } \kappa = 0 \\ R_0 \sinh \left( \frac{r'}{R_0} \right) & \text{when } \kappa = -1 \end{cases}$$

Dropping the prime notation for  $r$ , the metric takes a form which is very similar to equation (29) but now with the time dependent expansion of space isolated in a separate factor. It must be emphasized that contrary to before,  $r$  is now a comoving coordinate.

$$dl^2 = a^2(t) \left[ dr^2 + S_\kappa^2(r) d\Omega^2 \right]$$

$$\text{with } d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

$$\text{and } S_\kappa(r) = \begin{cases} R_0 \sin \left( \frac{r}{R_0} \right) & \text{when } \kappa = +1 \\ r & \text{when } \kappa = 0 \\ R_0 \sinh \left( \frac{r}{R_0} \right) & \text{when } \kappa = -1 \end{cases}$$

## 4.2 The FLRW metric

A spacetime metric incorporates the time coordinate on par with the spatial coordinates and gives the infinitesimal separation  $ds$  between 2 events in spacetime instead of the infinitesimal distance  $dl$  between 2 point in space.

If the cosmological principle<sup>2</sup> is accepted as valid, the spacetime metric has to be everywhere the same and  $a(t)$  can only depend on time. The simplest metric for a homogeneous and isotropic spacetime is then:

$$\begin{aligned}
 ds^2 &= c^2 dt^2 - a^2(t) [dr^2 + S_\kappa^2(r) d\Omega^2] \\
 \text{with } d\Omega^2 &= d\theta^2 + \sin^2 \theta d\phi^2 \\
 \text{and } S_\kappa(r) &= \begin{cases} R_0 \sin\left(\frac{r}{R_0}\right) & \text{when } \kappa = +1 \\ r & \text{when } \kappa = 0 \\ R_0 \sinh\left(\frac{r}{R_0}\right) & \text{when } \kappa = -1 \end{cases}
 \end{aligned} \tag{31}$$

This metric is commonly known as the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Alternatively, using a different coordinate system, the FLRW metric takes the form:

$$\begin{aligned}
 ds^2 &= c^2 dt^2 - a^2(t) \left[ \frac{d\rho^2}{1 - (\kappa/R_0^2) \rho^2} + \rho^2 d\Omega^2 \right] \\
 \text{with } d\Omega^2 &= d\theta^2 + \sin^2 \theta d\phi^2
 \end{aligned} \tag{32}$$

In the appendix it is shown that equation (32) transforms into equation (31) under the coordinate transformation:

$$\rho \equiv S_\kappa(r)$$

Depending on the sign of  $ds^2$ , there are 3 possibilities for the spacetime separation between events:

**space-like** The spacetime separation between 2 events is called "space-like" when  $ds^2 < 0$ , i.e. when the spatial term of the spacetime metric is dominant.

**time-like** The spacetime separation between 2 events is called "time-like" when  $ds^2 > 0$ , i.e. when the temporal term of the spacetime metric is dominant.

**light-like** The spacetime separation between 2 events is called "light-like" when  $ds^2 = 0$  as light always follows a path for which  $ds^2 = 0$ .

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<sup>2</sup>The cosmological principle states that the universe is homogeneous and isotropic, i.e. that it has the same appearance and properties at every location and in every direction.

To illustrate the above, depict 2 light bulbs which simultaneously emit a flash of light. Both flashes occur with a space-like separation given that  $dt = 0$  and that both light bulbs can obviously not be at the same location. Similarly, depict a single light bulb which emits 2 flashes at an interval  $dt$ . If the light bulb remains stationary between the flashes, they now occur with a time-like separation given that  $dt \neq 0$ .

A spacetime metric in terms of 4 freely chosen coordinates  $(x^0, x^1, x^2, x^3)$  and written in a very general way as a double summation has the form:

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu \quad (33)$$

The factors  $g_{\mu\nu}$  are the components of the metric tensor in Einstein's field equations of general relativity:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (34)$$

In equation (34),  $R_{\mu\nu}$  is the Ricci tensor,  $g_{\mu\nu}$  the metric tensor and  $T_{\mu\nu}$  the stress-energy-momentum tensor. The factor  $R$  is the scalar curvature,  $\Lambda$  the cosmological constant,  $G$  the gravitational constant and  $c$  the speed of light. Tensors are a generalization of vectors and follow a particular calculus which is a field of study on its own. To keep things simple, consider the tensors in Einstein's field equation to be symmetrical 4x4 matrices which implies that equation (34) actually represents 10 independent equations.

The Ricci tensor  $R_{\mu\nu}$  is a function of first and second order derivatives of  $g_{\mu\nu}$ . The scalar curvature  $R$  is related to  $R_{\mu\nu}$  and therefore also dependent on first and second order derivatives of  $g_{\mu\nu}$ .

If the cosmological principle is accepted as valid, the stress-energy-momentum tensor  $T_{\mu\nu}$  is a diagonal tensor, i.e. a tensor whose elements are zero except those on the main diagonal. If it is further assumed that the content of the universe on a sufficiently large scale behaves like a fluid<sup>3</sup>, the stress-energy-momentum tensor takes the following form in which  $\epsilon$  is the energy density and  $P$  the pressure of the fluid:

$$T_{\mu\nu} = \begin{bmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

It should be understandable that combining equations (32), (33) and (34) and introducing the dot notation for time derivatives leads to the following differential equations for the scale factor  $a(t)$ :

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<sup>3</sup>A fluid is a substance which does not resist deformation when a force is applied to it. Fluids are subdivided in incompressible fluids like liquids and compressible fluids like gases and plasmas.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \epsilon - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2} (\epsilon + 3P) + \frac{\Lambda}{3}$$

The first equation is known as the Friedmann<sup>4</sup> equation, named after the Russian cosmologist who first derived it from Einstein's field equations. The second equation is known as the acceleration equation and tells something about how rapidly  $a(t)$  changes over time.

In some textbooks, the factor  $\kappa/R_0^2$  in the Friedmann equation is replaced by an alternative curvature parameter  $k$  defined as:

$$k \equiv \frac{\kappa}{R_0^2}$$

Obviously,  $k$  retains the sign of  $\kappa$  but is no longer limited to +1, 0 or -1 and takes any value as determined by the value of  $R_0$ .

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<sup>4</sup>Alexander Friedmann was a Russian physicist and mathematician who lived from 1888 till 1925.

## A Equivalence of FLRW metric notations

The Friedmann-Lemaître-Robertson-Walker spacetime metric takes different shapes depending on the coordinate system which is used. One form makes use of the function  $S_\kappa(r)$  as shown in notation (31) while another form contains coordinate  $\rho$  instead of function  $S_\kappa(r)$  as shown in notation (32). Although they look substantially different, one transforms into the other under the coordinate transformation:

$$\rho \equiv S_\kappa(r)$$

For a positive curvature ( $\kappa = +1$ ):

$$\rho = S_\kappa(r) = R_0 \sin\left(\frac{r}{R_0}\right)$$

Consequently, the derivative  $d\rho$  becomes:

$$d\rho = R_0 \cos\left(\frac{r}{R_0}\right) \frac{dr}{R_0} = \cos\left(\frac{r}{R_0}\right) dr$$

Using the above expressions for  $\rho$  and  $d\rho$ , the first term between square brackets in metric (32) simplifies to:

$$\frac{d\rho^2}{1 - (\kappa/R_0^2) \rho^2} = \frac{\cos^2\left(\frac{r}{R_0}\right)}{1 - \sin^2\left(\frac{r}{R_0}\right)} dr^2 = dr^2$$

In the absence of curvature ( $\kappa = 0$ ):

$$\rho = S_\kappa(r) = r$$

Consequently, the derivative  $d\rho$  becomes:

$$d\rho = dr$$

Using the above expressions for  $\rho$  and  $d\rho$ , the first term between square brackets in metric (32) again simplifies to:

$$\frac{d\rho^2}{1 - (\kappa/R_0^2) \rho^2} = dr^2$$

For a negative curvature ( $\kappa = -1$ ):

$$\rho = S_{\kappa}(r) = R_0 \sinh\left(\frac{r}{R_0}\right)$$

Consequently, the derivative  $d\rho$  becomes:

$$d\rho = R_0 \cosh\left(\frac{r}{R_0}\right) \frac{dr}{R_0} = \cosh\left(\frac{r}{R_0}\right) dr$$

Using the above expressions for  $\rho$  and  $d\rho$ , the first term between square brackets in metric (32) simplifies once more to:

$$\frac{d\rho^2}{1 - (\kappa/R_0^2) \rho^2} = \frac{\cosh^2\left(\frac{r}{R_0}\right)}{1 + \sinh^2\left(\frac{r}{R_0}\right)} dr^2 = dr^2$$

The conclusion is that for all values of  $\kappa$ , metrics (31) and (32) are fully equivalent:

$$ds^2 = c^2 dt^2 - a^2(t) \left[ \frac{d\rho^2}{1 - (\kappa/R_0^2) \rho^2} + \rho^2 d\Omega^2 \right]$$

$$ds^2 = c^2 dt^2 - a^2(t) [dr^2 + \rho^2 d\Omega^2]$$

$$ds^2 = c^2 dt^2 - a^2(t) [dr^2 + S_{\kappa}^2(r) d\Omega^2]$$